*Research Inventy: International Journal of Engineering And Science*
*Vol.12, Issue 4 (April 2022), PP 18-23*
*Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com*

# A Study on Arc Length of Nondifferentiable Curves

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***Abstract:*** *In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we obtain three arc length formulas of fractional differentiable plane curves in the form of parametric equation, fractional analytic function, and polar coordinate equation. These formulas are generalizations of traditional arc length formulas of differentiable plane curves. The main methods used in this paper are the product rule and the chain rule for fractional derivatives. A new multiplication of fractional analytic functions plays an important role in this study. On the other hand, some examples are given to illustrate our results.*

***Keywords:*** *Jumarie type of R-L fractional calculus, Arc length formulas, Fractional differentiable plane curves, Product rule, Chain rule, New multiplication, Fractional analytic functions.*

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Date of Submission: 04-04-2022 Date of Acceptance: 19-04-2022

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## I. INTRODUCTION

In recent years, fractional calculus has been widely used in almost all branches of sciences and engineering such as mechanics, dynamics, control theory, electrical engineering, economics, and physics [1-10]. Fractional calculus is a generalization of traditional calculus of integer order, but it is different from classical calculus. There is no unique definition of fractional derivative and integral. Commonly used definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [1-3]. In addition, the application of fractional calculus can be referred to [4-15].

In this paper, the arc length problem of fractional differentiable plane curve is studied. Based on Jumarie′s modified R-L fractional calculus and a new multiplication of fractional analytic functions, we can obtain three arc length formulas of fractional plane curves in the form of parametric equation, fractional analytic function and polar coordinate equation. In fact, the formulas we obtained are generalizations of the arc length formulas of classical differentiable plane curves. Furthermore, the new multiplication is a natural opetation of fractional analytic functions, and it plays an important role in this article. In addition, we give three examples to illustrate our results.

## II. PRELIMINARIES

In the following, we introduce the fractional calculus used in this article.

**Definition 2.1** ([16])**:** Assume that $0 < \alpha < 1$, and $t_0$ is a real number. Then the Jumarie type of Riemann-Liouville $\alpha$-fractional derivative is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{f(x)-f(a)}{(t-x)^{\alpha}}dx\ . \tag{1}$$

And the Jumarie type of Riemann-Liouville $\alpha$-fractional integral is defined by

$$\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(x)}{(t-x)^{1-\alpha}}dx\ . \tag{2}$$

Where $\Gamma(\ \ )$ is the gamma function.

**Proposition 2.2** ([17])**:** *If $t_0, \alpha, \beta, C$ are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{t_0}D_t^{\alpha}\right)\left[(t-t_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha}, \tag{3}$$

*and*

$$\left({}_{t_0}D_t^{\alpha}\right)[C] = 0. \tag{4}$$

The following is the definition of fractional analytic function.

**Definition 2.3** ([18])**:** If $t, t_0$, and $a_k$ are real numbers for all $k$, $t_0 \in (a,b)$, and $0 < \alpha < 1$. If the function $f_{\alpha}: [a,b] \to R$ can be expressed as an $\alpha$-fractional power series, i.e., $f_{\alpha}((t-t_0)^{\alpha}) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$ on some open interval $(t_0 - r, t_0 + r)$, then we say that $f_{\alpha}((t-t_0)^{\alpha})$ is $\alpha$-fractional analytic at $t_0$, where $r$ is the radius of convergence about $t_0$. Moreover, if $f_{\alpha}: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is $\alpha$-fractional analytic at every point in open interval $(a,b)$, then we say that $f_{\alpha}$ is an $\alpha$-fractional analytic function on $[a,b]$.

Next, we define a new multiplication of fractional analytic functions.

**Definition 2.4** ([18])**:** Suppose that $0 < \alpha < 1$, and $t_0$ is a real number. If $f_\alpha((t-t_0)^\alpha)$ and $g_\alpha((t-t_0)^\alpha)$ are two $\alpha$-fractional analytic functions on an interval containing $t_0$,

$$f_\alpha((t-t_0)^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}, \tag{5}$$

$$g_\alpha((t-t_0)^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}. \tag{6}$$

Then we define

$$\begin{aligned}&f_\alpha((t-t_0)^\alpha)\otimes g_\alpha((t-t_0)^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\otimes\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}\\&=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(t-t_0)^{k\alpha}.\end{aligned} \tag{7}$$

In other words,

$$\begin{aligned}&f_\alpha((t-t_0)^\alpha)\otimes g_\alpha((t-t_0)^\alpha)\\&=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}\otimes\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}\\&=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}.\end{aligned} \tag{8}$$

**Definition 2.5** ([18])**:** Let $0 < \alpha < 1, t_0$ be a real number, and $f_\alpha((t-t_0)^\alpha)$, $g_\alpha((t-t_0)^\alpha)$ be $\alpha$-fractional analytic functions defined on an interval containing $t_0$,

$$f_\alpha((t-t_0)^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}, \tag{9}$$

$$g_\alpha((t-t_0)^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}=\sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha((t-t_0)^\alpha)$ and $g_\alpha((t-t_0)^\alpha)$ are defined as follows:

$$(f_\alpha\circ g_\alpha)((t-t_0)^\alpha)=f_\alpha\big(g_\alpha((t-t_0)^\alpha)\big)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\big(g_\alpha((t-t_0)^\alpha)\big)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)((t-t_0)^\alpha)=g_\alpha\big(f_\alpha((t-t_0)^\alpha)\big)=\sum_{k=0}^{\infty}\frac{b_k}{k!}\big(f_\alpha((t-t_0)^\alpha)\big)^{\otimes k}. \tag{12}$$

**Definition 2.6** ([18])**:** Suppose that $0 < \alpha < 1$, $t_0$ is a real number. If $f_\alpha((t-t_0)^\alpha)$, $g_\alpha((t-t_0)^\alpha)$ are two $\alpha$-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)((t-t_0)^\alpha)=(g_\alpha\circ f_\alpha)((t-t_0)^\alpha)=\frac{1}{\Gamma(\alpha+1)}(t-t_0)^\alpha. \tag{13}$$

Then we say these two fractional analytic functions are inverse to each other.

**Theorem 2.7** (product rule for fractional derivatives) ([20])**:** *Let* $0 < \alpha < 1, t_0$ *be a real number, and let* $f_\alpha((t-t_0)^\alpha), g_\alpha((t-t_0)^\alpha)$ *be $\alpha$-fractional analytic functions defined on an interval containing* $t_0$. *Then*

$$\begin{aligned}&\left({}_{t_0}D_t^\alpha\right)[\,f_\alpha((t-t_0)^\alpha)\otimes g_\alpha((t-t_0)^\alpha)]\\&=\left({}_{t_0}D_t^\alpha\right)[f_\alpha((t-t_0)^\alpha)]\otimes g_\alpha((t-t_0)^\alpha)+f_\alpha((t-t_0)^\alpha)\otimes\left({}_{t_0}D_t^\alpha\right)[g_\alpha((t-t_0)^\alpha)].\end{aligned} \tag{14}$$

The followings are some fractional analytic functions.

**Definition 2.8** ([20])**:** Assume that $0 < \alpha < 1$ , and $t$ is a real variable. The $\alpha$-fractional exponential function is defined by

$$E_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes k}. \tag{15}$$

And the $\alpha$-fractional logarithmic function $Ln_\alpha(t^\alpha)$ is the inverse function of the $E_\alpha(t^\alpha)$**.** On the other hand, the $\alpha$-fractional sine and cosine function are defined as follows:

$$sin_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{16}$$

and

$$cos_\alpha(t^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k\alpha}}{\Gamma(2k\alpha+1)}. \tag{17}$$

**Proposition 2.9** (fractional Euler's formula)[17]**:** *Let* $0 < \alpha < 1$, *then*

$$E_\alpha(it^\alpha)=cos_\alpha(t^\alpha)+isin_\alpha(t^\alpha). \tag{18}$$

**Remark 2.10:** If $\alpha = 1$, then we obtain the classical Euler's formula $e^{it} = cost + isint$. On the other hand, the smallest positive real number $T_\alpha$ such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(it^\alpha)$.

**Definition 2.11:** Let $0 < \alpha < 1$, and $r, t_0$ be real numbers. The $r$-th power of the $\alpha$-fractional analytic function $f_\alpha((t-t_0)^\alpha)$ is defined by $[\,f_\alpha((t-t_0)^\alpha)]^{\otimes r}=E_\alpha\left(rLn_\alpha\big(\,f_\alpha((t-t_0)^\alpha)\big)\right)$. On the other hand, if $g_\alpha((t-t_0)^\alpha)$ is also an $\alpha$-fractional analytic function such that $f_\alpha((t-t_0)^\alpha)\otimes g_\alpha((t-t_0)^\alpha)=1$, then we say that $g_\alpha((t-t_0)^\alpha)$ is the $\otimes$ reciprocal of $f_\alpha((t-t_0)^\alpha)$, and is denoted by $[f_\alpha((t-t_0)^\alpha)]^{\otimes -1}$.

**Theorem 2.12** ([20])**:** *Let* $0 < \alpha < 1$, *and* $t_0$ *be a real number. Then*

$$\left({}_{t_0}D_t^{\alpha}\right)[\,sin_{\alpha}((t-t_0)^{\alpha})] = cos_{\alpha}((t-t_0)^{\alpha}), \tag{19}$$

$$\left({}_{t_0}D_t^{\alpha}\right)[\,cos_{\alpha}((t-t_0)^{\alpha})] = -sin_{\alpha}((t-t_0)^{\alpha}), \tag{20}$$

**Theorem 2.13** (chain rule for fractional derivatives) ([19])**:** *If* $0<\alpha<1$, *and let* $f_{\alpha}((t-t_0)^{\alpha}), g_{\alpha}((t-t_0)^{\alpha})$ *be* $\alpha$*-fractional analytic functions. Then*

$$\left({}_{t_0}D_t^{\alpha}\right)[\,f_{\alpha}(g_{\alpha}((t-t_0)^{\alpha}))] = \left({}_{t_0}D_t^{\alpha}\right)[f_{\alpha}((t-t_0)^{\alpha})](g_{\alpha}((t-t_0)^{\alpha})) \otimes \left({}_{t_0}D_t^{\alpha}\right)[g_{\alpha}((t-t_0)^{\alpha})]. \tag{21}$$

## III. RESULTS AND EXAMPLES

First, the arc length definition of fractional differentiable plane curve in the form of parametric equation is introduced.

**Definition 3.1:** Suppose that $0<\alpha<1$, $\alpha$ is a real number, and $x_{\alpha}((t-a)^{\alpha}), y_{\alpha}((t-a)^{\alpha})$ are $\alpha$-fractional analytic functions defined on an interval containing $[a,b]$. Let the parametric equation of an $\alpha$-fractional differentiable plane curve be $\begin{cases} x_{\alpha} = x_{\alpha}((t-a)^{\alpha}) \\ y_{\alpha} = y_{\alpha}((t-a)^{\alpha}) \end{cases}$. Then the arc length of this $\alpha$-fractional differentiable plane curve is

$$s_{\alpha} = \left({}_{a}I_b^{\alpha}\right)\left[\left[\left[\left({}_{a}D_t^{\alpha}\right)[x_{\alpha}((t-a)^{\alpha})]\right]^{\otimes 2} + \left[\left({}_{a}D_t^{\alpha}\right)[y_{\alpha}((t-a)^{\alpha})]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]. \tag{22}$$

**Remark 3.2:** If $\alpha=1$, then Eq. (22) becomes the traditional arc length formula of differentiable plane curves.

Secondly, the arc length formula of fractional differentiable plane curve in the form of fractional analytical function is introduced.

**Theorem 3.3:** *Let* $0<\alpha<1$. *If* $y_{\alpha}((x-a)^{\alpha})$ *is an* $\alpha$*-fractional analytic function defined on an interval containing* $[a,b]$. *Then the arc length of the* $\alpha$*-fractional differentiable plane curve* $y_{\alpha} = y_{\alpha}((x-a)^{\alpha})$ *is*

$$s_{\alpha} = \left({}_{a}I_b^{\alpha}\right)\left[\left[1 + \left[\left({}_{a}D_x^{\alpha}\right)[y_{\alpha}((x-a)^{\alpha})]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]. \tag{23}$$

**Proof** Since the parametric equation of this $\alpha$-fractional differentiable plane curve is $\begin{cases} x_{\alpha} = \frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha} \\ y_{\alpha} = y_{\alpha}((x-a)^{\alpha}) \end{cases}$, it follows from Definition 3.1 that its arc length is

$$s_{\alpha} = \left({}_{a}I_b^{\alpha}\right)\left[\left[\left[\left({}_{a}D_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}\right]\right]^{\otimes 2} + \left[\left({}_{a}D_x^{\alpha}\right)[y_{\alpha}((x-a)^{\alpha})]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$

$$= \left({}_{a}I_b^{\alpha}\right)\left[\left[1 + \left[\left({}_{a}D_x^{\alpha}\right)[y_{\alpha}((x-a)^{\alpha})]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right].$$

Q.e.d.

In the following, we introduce the arc length formula of fractional differentiable plane curve in the form of polar coordinate equation.

**Theorem 3.4:** *Assume that* $0<\alpha<1$, *and if* $\rho_{\alpha} = \rho_{\alpha}((\theta-a)^{\alpha})$ *is an* $\alpha$*-fractional analytic function defined on an interval containing* $[a,b]$, *and it is the polar coordinate equation of an* $\alpha$*-fractional differentiable plane curve. Then the arc length of this* $\alpha$*-fractional differentiable plane curve is*

$$s_{\alpha} = \left({}_{a}I_b^{\alpha}\right)\left[\left[[\rho_{\alpha}((\theta-a)^{\alpha})]^{\otimes 2} + \left[\left({}_{a}D_{\theta}^{\alpha}\right)[\rho_{\alpha}((\theta-a)^{\alpha})]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]. \tag{24}$$

**Proof** Since $\begin{cases} x_{\alpha}((\theta-a)^{\alpha}) = \rho_{\alpha}((\theta-a)^{\alpha}) \otimes cos_{\alpha}((\theta-a)^{\alpha}) \\ y_{\alpha}((\theta-a)^{\alpha}) = \rho_{\alpha}((\theta-a)^{\alpha}) \otimes sin_{\alpha}((\theta-a)^{\alpha}) \end{cases}$, it follows from the product rule for fractional derivatives that

$$\left[\left({}_{a}D_{\theta}^{\alpha}\right)[x_{\alpha}((\theta-a)^{\alpha})]\right]^{\otimes 2}$$

$$= \left[\left({}_{a}D_{\theta}^{\alpha}\right)[\rho_{\alpha}((\theta-a)^{\alpha}) \otimes cos_{\alpha}((\theta-a)^{\alpha})]\right]^{\otimes 2}$$

$$= \left[\left({}_{a}D_{\theta}^{\alpha}\right)[\rho_{\alpha}((\theta-a)^{\alpha})] \otimes cos_{\alpha}((\theta-a)^{\alpha}) - \rho_{\alpha}((\theta-a)^{\alpha}) \otimes sin_{\alpha}((\theta-a)^{\alpha})\right]^{\otimes 2}$$

$$= \left({}_{a}D_{\theta}^{\alpha}\right)^{2}[\rho_{\alpha}((\theta-a)^{\alpha})] \otimes [cos_{\alpha}((\theta-a)^{\alpha})]^{\otimes 2} + [\rho_{\alpha}((\theta-a)^{\alpha})]^{\otimes 2}[sin_{\alpha}((\theta-a)^{\alpha})]^{\otimes 2}$$

$$-2 \cdot sin_{\alpha}((\theta-a)^{\alpha}) \otimes cos_{\alpha}((\theta-a)^{\alpha}) \otimes \rho_{\alpha}((\theta-a)^{\alpha}) \otimes \left({}_{a}D_{\theta}^{\alpha}\right)[\rho_{\alpha}((\theta-a)^{\alpha})]. \tag{25}$$

And

$$\left[\left({}_{a}D_{\theta}^{\alpha}\right)[y_{\alpha}((\theta-a)^{\alpha})]\right]^{\otimes 2}$$

$$=\left[\left({}_aD_\theta^\alpha\right)[\rho_\alpha((\theta-a)^\alpha)\otimes sin_\alpha((\theta-a)^\alpha)]\right]^{\otimes 2}$$
$$=\left[\left({}_aD_\theta^\alpha\right)[\rho_\alpha((\theta-a)^\alpha)]\otimes sin_\alpha((\theta-a)^\alpha)+\rho_\alpha((\theta-a)^\alpha)\otimes cos_\alpha((\theta-a)^\alpha)\right]^{\otimes 2}$$
$$=\left({}_aD_\theta^\alpha\right)^2[\rho_\alpha((\theta-a)^\alpha)]\otimes[sin_\alpha((\theta-a)^\alpha)]^{\otimes 2}+[\rho_\alpha((\theta-a)^\alpha)]^{\otimes 2}[cos_\alpha((\theta-a)^\alpha)]^{\otimes 2}$$
$$+2\cdot sin_\alpha((\theta-a)^\alpha)\otimes cos_\alpha((\theta-a)^\alpha)\otimes\rho_\alpha((\theta-a)^\alpha)\otimes\left({}_aD_\theta^\alpha\right)[\rho_\alpha((\theta-a)^\alpha)]\ . \tag{26}$$

Therefore,

$$\left[\left({}_aD_\theta^\alpha\right)[x_\alpha((\theta-a)^\alpha)]\right]^{\otimes 2}+\left[\left({}_aD_\theta^\alpha\right)[y_\alpha((\theta-a)^\alpha)]\right]^{\otimes 2}=[\rho_\alpha((\theta-a)^\alpha)]^{\otimes 2}+\left({}_aD_\theta^\alpha\right)^2[\rho_\alpha((\theta-a)^\alpha)]. \tag{27}$$

And hence, by Definition 3.1, the arc length of this $\alpha$-fractional differentiable plane curve is

$$s_\alpha=\left({}_aI_b^\alpha\right)\left[\left[\left[\left({}_aD_\theta^\alpha\right)[x_\alpha((\theta-a)^\alpha)]\right]^{\otimes 2}+\left[\left({}_aD_\theta^\alpha\right)[y_\alpha((\theta-a)^\alpha)]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=\left({}_aI_b^\alpha\right)\left[\left[[\rho_\alpha((\theta-a)^\alpha)]^{\otimes 2}+\left[\left({}_aD_\theta^\alpha\right)[\rho_\alpha((\theta-a)^\alpha)]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]. \tag{28}$$

**Example 3.5:** Suppose that $0<\alpha<1$. Find the arc length of the $\alpha$-fractional differentiable plane curve $y_\alpha(x^\alpha)=Ln_\alpha\left(cos_\alpha(x^\alpha)\right)$ from $x=0$ to $x=\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}$.

**Solution** By Theorem 3.3 and the chain rule for fractional derivatives, we obtain the arc length of this $\alpha$-fractional differentiable plane curve

$$s_\alpha=\left({}_0I^\alpha_{\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}}\right)\left[\left[1+\left[\left({}_0D_x^\alpha\right)\left[Ln_\alpha\left(cos_\alpha(x^\alpha)\right)\right]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=\left({}_0I^\alpha_{\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}}\right)\left[[1+[-tan_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=\left({}_0I^\alpha_{\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}}\right)\left[[[sec_\alpha(x^\alpha)]^{\otimes 2}]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=\left({}_0I^\alpha_{\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}}\right)[sec_\alpha(x^\alpha)]$$
$$=Ln_\alpha\left(sec_\alpha(x^\alpha)+tan_\alpha(x^\alpha)\right)\Big|_0^{\left(\frac{T_\alpha}{8}\right)^{\frac{1}{\alpha}}}$$
$$=Ln_\alpha\left(sec_\alpha\left(\frac{T_\alpha}{8}\right)+tan_\alpha\left(\frac{T_\alpha}{8}\right)\right)-Ln_\alpha\left(sec_\alpha(0)+tan_\alpha(0)\right)$$
$$=Ln_\alpha\left(sec_\alpha\left(\frac{T_\alpha}{8}\right)+tan_\alpha\left(\frac{T_\alpha}{8}\right)\right). \tag{29}$$

**Example 3.6:** Suppose that $0<\alpha<1$ and $p>0$. Find the arc length of the $\alpha$-fractional cycloid

$$\begin{cases} x_\alpha(\theta^\alpha)=p\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha-sin_\alpha(\theta^\alpha)\right)\\ y_\alpha(\theta^\alpha)=p\left(1-cos_\alpha(\theta^\alpha)\right)\end{cases} \tag{30}$$

from $\theta=0$ to $\theta=(T_\alpha)^{\frac{1}{\alpha}}$.

**Solution** Using Definition 3.1 yields the arc length of this $\alpha$-fractional cycloid

$$s_\alpha=\left({}_0I^\alpha_{(T_\alpha)^{\frac{1}{\alpha}}}\right)\left[\left[\left[\left({}_0D_\theta^\alpha\right)\left[p\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha-sin_\alpha(\theta^\alpha)\right)\right]\right]^{\otimes 2}+\left[\left({}_0D_\theta^\alpha\right)\left[p\left(1-cos_\alpha(\theta^\alpha)\right)\right]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=\left({}_0I^\alpha_{(T_\alpha)^{\frac{1}{\alpha}}}\right)\left[\left[\left[p\left(1-cos_\alpha(\theta^\alpha)\right)\right]^{\otimes 2}+[p\cdot sin_\alpha(\theta^\alpha)]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=p\cdot\left({}_0I^\alpha_{(T_\alpha)^{\frac{1}{\alpha}}}\right)\left[[2-2cos_\alpha(\theta^\alpha)]^{\otimes\left(\frac{1}{2}\right)}\right]$$
$$=2p\cdot\left({}_0I^\alpha_{(T_\alpha)^{\frac{1}{\alpha}}}\right)\left[sin_\alpha\left(\frac{1}{2}\theta^\alpha\right)\right]$$

$$= 2p \cdot (-2) \cdot cos_\alpha\left(\frac{1}{2}\theta^\alpha\right)\Big]_0^{(T_\alpha)^{\frac{1}{\alpha}}}$$

$$= 4p\left(1 - cos_\alpha\left(\tfrac{1}{2}T_\alpha\right)\right). \tag{31}$$

**Example 3.7:** If $0 < \alpha < 1$, $p > 0$. Find the arc length of the $\alpha$-fractional cardioid $\rho_\alpha(\theta^\alpha) = p\left(1 + cos_\alpha(\theta^\alpha)\right)$.

**Solution** By Theorem 3.4, we obtain the arc length of this $\alpha$-fractional cardioid is

$$s_\alpha = 2\left({}_0I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}}\right)\left[\left[\left[p\left(1 + cos_\alpha(\theta^\alpha)\right)\right]^{\otimes 2} + \left[\left({}_aD^\alpha_\theta\right)\left[p\left(1 + cos_\alpha(\theta^\alpha)\right)\right]\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right].$$

$$= 2\left({}_0I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}}\right)\left[\left[\left[p\left(1 + cos_\alpha(\theta^\alpha)\right)\right]^{\otimes 2} + \left[-p \cdot sin_\alpha(\theta^\alpha)\right]^{\otimes 2}\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$

$$= 2p \cdot \left({}_0I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}}\right)\left[\left[2 + 2cos_\alpha(\theta^\alpha)\right]^{\otimes\left(\frac{1}{2}\right)}\right]$$

$$= 4p \cdot \left({}_0I^{\alpha}_{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}}\right)\left[cos_\alpha\left(\frac{1}{2}\theta^\alpha\right)\right]$$

$$= 8p \cdot sin_\alpha\left(\frac{1}{2}\theta^\alpha\right)\Big]_0^{\left(\frac{T_\alpha}{2}\right)^{\frac{1}{\alpha}}}$$

$$= 8p \cdot sin_\alpha\left(\frac{T_\alpha}{4}\right). \tag{32}$$

## IV. CONCLUSION

Based on Jumarie′s modified R-L fractional calculus and a new multiplication of fractional analytic functions, three types of arc length formulas of fractional differentiable plane curves can be obtained. The major methods used in this paper are the product rule and the chain rule for fractional derivatives. In fact, the new multiplication is a natural operation of fractional analytic functions, and these formulas are generalizations of classical arc length formulas of differentiable plane curves. The new multiplication concept of fractional analytic functions plays an important role in this paper. In the future, we will use the new multiplication to expand the research fields to engineering mathematics and fractional differential equations.

**Conflict of interest**

There is no conflict to disclose.

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